ACTEX Learning

Introduction to the Mathematics of Demography

Fourth Edition

4th

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Robert Bourbeau, PhD, Professor Emeritus of Demography

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PREFACE

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And

ROBERT BOURBEAU, PhD, Professor Emeritus of Demography

Robert Brown

This textbook first appeared in 1991. It started as a formal presentation of the lecture notes of Rob Brown who was then teaching a course in the Mathematics of Demography within the Actuarial Science program at the University of Waterloo.

At that time, the Society of Actuaries had an optional exam on the Mathematics of Demography using a textbook, *Demography Through Problems* (1977), co-authored by N. Keyfitz and J. A. Beekman. This textbook was written as a series of problems presented to students with solutions. But there was very little further discourse, so the “Course” was more or less, self-taught.

The first Edition of this text solved this “problem” by having nearly 300 pages of descriptive material, solved Examples, plus a long list of problems (with a Solution manual available). The 1st edition quickly became the recognized reference for Course 161, and we evolved from there.

The 2nd and 3rd editions were small updates and editorial improvements to the 1st Edition, which was still the foundation of the material.

At the turn of the century, the Society of Actuaries dropped its "College Catalogue" approach to the syllabus and returned to a smaller number of larger exams courses and exams. Thus, the Mathematics of Demography ceased to exist as a stand-alone study, and the obvious need for the textbook disappeared.

But small sales continued.

In the fall of 2018, I attended the Society of Actuaries Annual Meeting. At that time, I met with leaders of the ACTEX team, and we discussed the possibility of bringing the textbook back to life. Given that I had not been close to the material for almost 20 years and given how much data sources and data reporting has changed, I was reluctant to be the "new" author. However, I was aware that a colleague of mine, Dr. Robert Bourbeau, was about to enter retirement. I immediately contacted him to see if he would like to take up the challenge of updating the textbook and creating a 4th edition.

Robert Bourbeau

I would like to thank Rob Brown for inviting me to become a co-author for this 4th Edition. Based on my background in Actuarial Science (BSc Université Laval) and my long-term experience as a teacher of Fundamentals of Demographic Analysis in the Department of Demography at Université de Montréal, I am more than happy to contribute to updating the textbook. I am very familiar with prior editions of this textbook. In fact, it was one of the main references for my students in demography and in actuarial science for my courses in Demographic Analysis and in Mathematics of Demography.

As a Demographer, I can add to this textbook my knowledge of the best practices in demographic analysis, including the importance of cohort analysis, to get a better picture of the phenomena responsible for population renewal.

It is important to state that this new edition covers, to a great extent, the same material in terms of presentation of the fundamentals of the mathematics of demography. However, as it has now been more than twenty years since the publication of the Third Edition of this text, the 4th Edition includes several important changes in content.

First, the collection of data has changed, and their quality has improved over the years. This change creates new methods for a better estimation of population counts and of components of population change (infant mortality rates, old age mortality patterns to give only two examples).

In Chapter 1, the description of census-taking in Canada and the United States now refers to the 2016 Canadian census and the 2010 U.S. census, with some details concerning the ongoing census in the U.S. (2020) and the upcoming census in Canada (2021). This updating echoes throughout the text, in that later numerical examples and exercises are evaluated using the most recent censuses and vital statistics data.

Chapter 2 presents more clearly the two main approaches in demography: cohort measures and period measures. This essential distinction is illustrated by the Lexis Diagram in a new format that is now the standard in demographic and actuarial publications. A method for decomposition of the difference between two crude rates, for two populations or for the same population at two points in time, is proposed in order to separate the effect of the differences in age structure and in death or fertility rates between the two populations. Additional estimations of infant mortality rate (IMR) are suggested to improve the monitoring of this important indicator. Finally, this chapter introduces the concept of translation of period measures into cohort measures for fertility indices, like the Total Fertility Rate (TFR) and the number of Children Ever Born (CEB).

Consistent with the updating of Chapter 1 for the most recent national population censuses or estimations and vital statistics, Chapter 4 now presents a description of the construction of the most recent national life tables, the 2017 U.S. Life Tables (replacing the 1979-81 version) and the 2015-2017 Canadian Life Tables (replacing the 1985-87 version). As Infant Life Tables are no longer published by the statistical agencies in Canada and the U.S., this topic has been deleted from Chapter 4. However, to reflect the changes in the age-at-death distribution, where more and more deaths are now occurring at old ages, special attention is given to estimation of old-age mortality trajectories in the construction of life tables.

There are no major changes to Chapters 5 and 6, except for two things. In Chapter 5, the presentation of Stationary Population Theory benefits from the use of the updated Lexis Diagrams, particularly through illustrative solutions in the examples. In Chapter 6, the most recent census and vital statistics data are used for the estimates of a population intrinsic growth rate.

Section 7.4, describing *Dynamic* methodologies for population projections, has been revised to reflect new trends in mortality and fertility. Classic mortality models (Gompertz, Makeham, etc.) are challenged by recent trends at old ages showing a deceleration of mortality rates and even a levelling off among supercentenarians. Fertility theories of Easterlin and Ermisch, published in the 80's, are confronted with new realities, showing a persistent low fertility level since the 90's.

Major changes in Chapter 8 are due to Rob Brown's expertise in Social Security issues. Population aging, recent changes in the financial markets and in public expenditures in health care are well addressed in this new edition.

Throughout the text, a considerable number of exercises has been updated with recent data added.

We hope the reader familiar with prior editions of this text will agree that significant improvements have been made in the new edition, both because of important new material and improved pedagogic presentation. This new Edition will also be useful to new undergraduate and graduate students in actuarial science, statistics, and demography, to be better equipped to analyze emerging socio-demographic issues during the 21st Century.

The authors would like to thank Ahmed Sarni, an undergraduate student in demography at the Université de Montréal, for his invaluable help in updating many tables and figures and in writing new equations in LaTex, a more flexible software.

January, 2022

CHAPTER 2

MEASURES OF MORTALITY AND FERTILITY

2.1 INTRODUCTION

Demographers adopt two complementary viewpoints when analyzing population data. The first approach is to determine, for example, how many years a person will live on average during his/her lifetime or how many children are born on average to a woman during her lifetime. The objective here is to relate demographic events to the individual life span, so this is called longitudinal analysis (or ***cohort analysis***). The other approach is to examine how a population changes from year to year, and what are the components of this change. Demographic events are studied on a yearly or a period basis, so this approach is called ***period analysis***. In this chapter, we present different measures of mortality and fertility by drawing from both approaches, and we highlight the differences that result from each viewpoint.

2.2 THE LEXIS DIAGRAM

One simple way to illustrate the two approaches mentioned above is to use the ***Lexis diagram*** [1], where one can represent demographic events

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[1] Wilhelm Lexis (1837-1914), a German statistician and demographer, was the first to introduce this diagram in 1875. The original diagram is slightly different from the modern version suggested by the French demographer Roland Pressat in 1959.

according to the three dimensions of time (calendar period), age and cohort (Figure 2.1).

2.2.1 Lexis Diagram: Cohort and Period Analysis

The following are the features of the Lexis diagram.

(1) A fixed point in time is represented by a vertical line.

(2) A fixed age is represented by a horizontal line.

(3) By intersecting a point in time and an age, one can identify individuals' "lifelines" along a 45° path, upward and to the right, since with each passing unit of time they increase in age by that same unit.

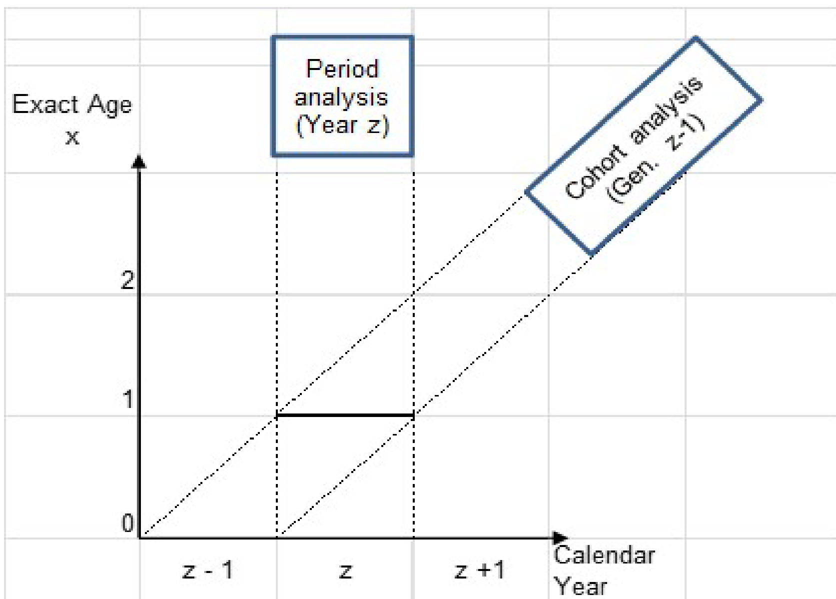

FIGURE 2.1 Lexis diagram: Cohort and period analysis

A person born on January 1 of year $z - 1$ will follow the upper diagonal and a person born on January 1 of year z the lower diagonal. All events concerning the $z - 1$ cohort will occur within the surface between these two lifelines.

Accordingly, all events occurring between January 1, *z* and January 1, *z* + 1, belong to year *z*.

2.2.2 Exact Age (Or Exact Duration) and Age at Last Birthday (Completed Duration)

According to Figure 2.1, all the members of a cohort who attain ***exact age*** 1 during year *z* would be on a horizontal line (in bold black on the Lexis diagram); members situated on a vertical line, for example, on January 1, *z* are identified by their ***age last birthday***, 0 for those between exact age 0 and exact age 1. Demographic events can also be identified by their age last birthday; for example, on Figure 2.1, all events occurring between exact age 0 and exact age 1 during year *z* (in a square) are classified as age 0 last birthday. Demographers often classify populations or events according to age last birthday for simplicity or when data are presented by age groups. For example, if population data are available by 5-year groups, from 0 to 5, from 5 to 10, etc., we use age last birthday 0-4, 5-9, etc. Each group contains 5 single years and can be noted as ${}_5P_x$. An age specific death rate for a 5-year group will be noted as ${}_5m_x$.

$$ {}_5m_x = \frac{{}_5D_x}{{}_5P_x} \, . \tag{2.1} $$

In Equation 2.1, deaths occur between exact age *x* and (*x* + 5) but do not include deaths at exact age (*x* + 5), which is equal to deaths at age *x*, *x* + 4 last birthday. We do the same thing for the population. To clarify this notion of interval, we are introducing the notation [*x*, *x* + *n*). This represents the age group from exact age *x* (*x*.000000) to just before age *x* + *n*. So the square bracket means greater than or equal to and the round bracket means less than (but not equal to).

Demographers use a number of statistics in their analysis of population dynamics. These statistics should be carefully understood and used consistently. Unfortunately, some statistics that are commonly used with colloquial acceptance are technically used incorrectly. For example, the net error ratio defined in Chapter 1 is improperly called the net error rate in various publications.

Confusion can exist among terms such as ratios, proportions, and rates.

Any two measures can be compared in the form of a ***ratio***. For example, assume that for every 100 births, 52 are male and 48 are female. The ratio of male to female births is 52 to 48, or 52/48. We could also state that the ratio of female to male births is 48 to 52. This ratio is also known as the sex ratio at birth (105), expressed as the number of male births for 100 females.

If the denominator of the comparative statistic is the total number of all possible occurrences, then we have a ***proportion***. For example, the proportion of male births out of total births is 52/100 and the proportion of female births is 48/100.

If the proportion is quoted as a measure per hundred, it is referred to as a percent. Hence the male birth proportion is 52% and the female birth proportion is 48%.

If the statistic is associated with a time interval, it can be quoted as a ***rate***. For example, if there are 100 births in a year, in a country with a midyear population of 7000, then the annual birth rate is 14.3 per 1000.

Unfortunately, many statistics commonly referred to as rates are not. For example, if 60% of the female population of a certain age group is in the labor force, this statistic is often referred to as the female labor force participation rate. In fact, this is a proportion and not a rate.

Similarly, the percentage of the population that is out of work and actively looking for work on a particular day is normally called the unemployment rate. This is, in fact, the unemployment proportion.

In this text, the colloquially accepted phrase will be adopted. Readers should be aware that if a technically incorrect phrase has become widely accepted, it will be found in publications that are otherwise meticulously accurate.

2.3 CRUDE RATES

Crude rates are normally defined as the ratio of the number of vital events which occurred in a defined population in one calendar year, to the size of the population at midyear. The midyear statistic is used to approximate the average size of the population over the year.

For example, in the United States in 2017 there were a total of 2.813 million deaths and a midyear population of 325.719 million. Then the ***crude death rate*** for 2017 is defined to be $\frac{2.813}{325.719} = .00864$, which is frequently expressed as 8.64 per 1000. In general, we will express the crude death rate for calendar year z as

$$d_c^z = \frac{D^z}{P(z)}, \tag{2.2}$$

where D^z is the number of deaths in calendar year z and $P(z)$ is the midyear population.

Similarly, the ***crude birth rate*** for calendar year z is defined as

$$b_c^z = \frac{B^z}{P(z)}, \tag{2.3}$$

where B^z denotes the number of live births in calendar year z. For example, there were 3.853 million live births in the United States in 2017, so the crude birth rate for that year was $\frac{3.853}{325.719} = .01183$, or 11.83 per 1000.

A word about notation is in order. In this text, we will use symbols such as B^z and D^z to denote measures taken over a calendar year, and symbols such as $P(z)$ to denote measures taken at a point of time within a calendar year. In the United States and in Canada, $P(z)$ arises from the midyear population estimates based on census count and adjustments for net undercount/overcount.

Another crude measure is the ***crude rate of natural increase***, defined as the crude birth rate less the crude death rate. Notationally, we have

$$r_c^z = b_c^z - d_c^z \tag{2.4}$$

The crude rate of natural increase gives a rough measure of the internal (i.e., without migration) rate of growth of a population.

In a similar fashion we can define a wide variety of crude rates such as ***crude marriage rates***, ***crude divorce rates***, ***crude immigration rates***, and so on. Crude rates should be used with care, if not outright skepticism, for reasons that will be presented in the following sections.

EXAMPLE 2.1

Given the following data, determine (a) the proportion of deaths that are male, (b) the percentage of deaths that are female, (c) the ratio of female to male deaths, and (d) the crude death rate.

Midyear Population	Deaths in Calendar Year *z*		
	Male	Female	Total
20.1 million	77,330	42,192	119,522

SOLUTION

(a) Proportion (male) $\frac{77,330}{119,522} = .647$

(b) Percentage (female) $\frac{42,192}{119,522} = 35.3\%$

(c) Ratio (female/male) $\frac{42,192}{77,330} = .546$

(d) Crude death rate (d_c^z) $\frac{119,522}{20,100,000} = .00595$ ❑

2.4 AGE-SPECIFIC MORTALITY RATES

As noted in the introduction, there are two approaches to study demographic phenomena: cohort and period analysis. It is very important to make this distinction when analyzing a phenomena that is subject to changes in the distribution of events over age or any other duration, if individuals may postpone or anticipate the occurrence of this event. For example, a woman can have her children sooner or later during her life, depending on many factors (individual, economic, or other). That means that fertility as measured on a period basis may not reflect the real behavior of a cohort at the end of its period of reproduction. In the case of mortality, such changes in the timing of deaths are not usual, so the measure on a period basis is more frequent, as shown here by the calculation of age-specific mortality rates for a given period. However, one should remember that, as global mortality decreases over time as it has been the case for two centuries now, the level of mortality will always be lower for a cohort than for a period.

Tables 2.1a and b on pages 28-29 provide data on ***age-specific mortality rates***, denoted ${}_n m_x$, for the states of Illinois and Florida. Each age-specific rate is calculated by dividing deaths by midyear population between ages x and $x + n$. The total number of deaths for the state of Illinois in 2017 was 109,721 and the total population on July 1, 2017 was 12,802,023, giving an overall crude death rate of 8.57 per 1000. Similarly, the total number of deaths for Florida was 203,636 and its total population was 20,984,400, for a crude death rate of 9.70 per 1000.

These crude death rates are surprisingly different (a 13% difference) if we compare the age-specific rates as listed, where Florida rates are much lower for each age group over 65. The reason for this variation is the significant difference in the underlying demographics of the two states, as seen in Tables 2.1a and b on the following pages.

Obviously, the crude death rate in Florida may not be indicative of a higher mortality profile but may only indicate that more elderly people live in Florida than in Illinois. Therefore, to compare the mortality profiles of two jurisdictions, we are well advised not to use their crude death rates. Section 2.5 provides two more accurate methods for comparing mortality rates between areas and one method of decomposition of the difference between two crude rates for two areas.

2.5 ADJUSTED MEASURES OF MORTALITY

Depending on the data available to the demographer, there are two standard methods of arriving at statistics that more fairly compare the level of mortality in two jurisdictions.

2.5.1 Direct Method of Adjustment

The more accurate of the two methods is called the ***direct method of adjustment*** and can be used only if age-specific mortality rates are available for the two jurisdictions. When this is the case, a standard population is then chosen. In the above example, a natural choice for the standard population would be the U.S. population in 2017, presented in Table 2.2 on page 30.

TABLE 2.1a

ILLINOIS				
Age Group $[x, x+n)$	Population July 1, 2017 (thousands)	Percent	Deaths in 2017	${}_n m_x \cdot 10^3$
0-5	773	6.0	1,055	1,36
5-15	1,617	12.6	227	0.14
15-25	1,702	13.3	1,393	0.82
25-35	1,776	13.9	2,279	1.28
35-45	1,639	12.8	2,803	1.71
45-55	1,683	13.1	6,084	3.62
55-65	1,666	13.0	14,042	8.43
65-75	1,117	8.7	20,061	17.96
75-85	563	4.4	25,679	45.65
85+	266	2.1	36,095	135.75
Total	12,802	100.0	109,718**	8.57

$n = 5$ for $x = 0$; $n = 10$ for $5 < x < 75$; $n = \infty$ for $x = 85$.
** Age not stated for 3 deaths

2.4 Age-Specific Mortality Rates; 2.5 Adjusted Measures of Mortality

2-3. Consider the following data for a community and a standard population.

Age	Community			Standard Population		
$[x, x+20)$	No. Alive July 1, z	Deaths in z	${}_n m_x^z$	No. Alive July 1, z	Deaths in z	${}_n m_x^z$
0-20	5,000	25	.005	700,000	700	.001
20-40	3,000	18	.006	500,000	1,000	.002
40-60	2,000	20	.010	400,000	1,600	.004
60-80	1,000	20	.020	100,000	1,000	.010
	11,000	83		1,700,000	4,300	

(a) Calculate the crude death rate for the community.

(b) Calculate the directly adjusted death rate.

(c) Calculate the indirectly adjusted death rate.

(d) Use the double standardization method to decompose the difference between the crude rates in the community and in the standard population into two parts: the difference in frequencies (death rates) and the difference of population structures. Briefly comment on the result.

2-4. Consider the following information.

Age	Country A		State X	
$[x, x+20)$	Census Population	Deaths	Census Population	Deaths
0-20	100,000	1,000	10,000	200
20-40	80,000	1,600	9,000	90
40-60	60,000	1,800	7,000	280
60-80	40,000	2,000	3,000	150
80+	20,000	2,000	1,000	150

Using the Country-A data as the standard population, calculate the indirectly adjusted death rate for State X.

2-4 .03292

2-5 .18730

2-6 Treatment A

2-7 .00971

2-8 .00600

2-9 10,236

2-10 $d_c^{One} = .00886$; $d_c^{Two} = .00983$; Population One

2-11 1.04017

2-12 .14772; .14444

2-13 .11818; .18636

2-14 .29834

2-15 .00448

2-16 (a) .01389 (b) .01000

2-17 . 09990

2-18 .01003

2-19 (d) only

2-20 (a) 1.62290 (b) .79217 (c) .78367

2-21 (a) 6.50000 (b) 3.56800

2-22 .00900

2-23 .14894

2-24 .00816

ABOUT THE AUTHORS

ROB BROWN, PhD, FCIA, FSA, ACAS

Rob retired from the University of Waterloo program in Actuarial Science in 2010 after 39 years of teaching and research. In that time, he wrote seven books and over sixty refereed papers. His research focus is the design of financial security programs in times of rapidly shifting demographics. Rob was also President of the Canadian Institute of Actuaries in 1990/91, President of the Society of Actuaries in 2000/01 and President of the International Actuarial Association in 2014, and Research Chair for the Ontario Expert Commission on Pensions in 2007/08. Rob now resides in Victoria, BC.

ROBERT BOURBEAU, PhD
Professor Emeritus of Demography

Robert retired from the Department of Demography at the Université de Montréal in 2015 after 37 years of teaching and research. He was the Department Chair from 2002 to 2010. Now as Professor Emeritus, he remains active in his research field, and he sometimes also continues to teach demographic courses. His main research interests lie in mortality and longevity, more precisely in the study of old-age mortality as well as the analysis of mortality by causes of deaths. His work was presented at numerous international scientific conferences, and he wrote more than 70 refereed papers published in major journals of Demography, Public Health, and Actuarial Science. Robert was also a World Health Organization Expert on Injury Prevention in 1990/95, and a consultant on mortality estimation for the Colombian Departamento Administrativo Nacional de Estadística in 2016. Since 2001, he has been a Member of the Advisory Committee on Demographic Statistics and Studies for Statistics Canada.

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